4.1 An Overview of the Area Problem

Name Group No.

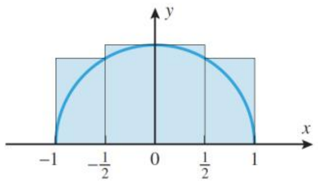
KEYWORD: method of exhaustion and rectangle method

QUICK CHECK EXERCISES

1. Let R denote the region below the graph of $f(x) = \sqrt{1 - x^2}$ and above the interval $[-1, 1]$.

(a) Use a geometric argument to find the area of R.

(b) What estimate results if the area of R is approximated by the total area within the rectangles of the accompanying figure?



◀ **Figure Ex-1**

2. Suppose that when the area A between the graph of a function $y = f(x)$ and an interval $[a, b]$ is approximated by the areas of n rectangles, the total area of the rectangles is $A_n = 2 + (2/n)$, $n = 1, 2, \ldots$. Then, $A =$ ________.

3. The area under the graph of $y = x^2$ over the interval $[0, 3]$ is ________.

4. Find a formula for the area $A(x)$ between the graph of the function $f(x) = x$ and the interval $[0, x]$, and verify that $A'(x) = f(x)$.

5. The area under the graph of $y = f(x)$ over the interval $[0, x]$ is $A(x) = x + e^x - 1$. It follows that $f(x) =$ ________.

True–False

Determine whether the statement is true or false. Explain your answer.

1. If $A(n)$ denotes the area of a regular n-sided polygon inscribed in a circle of radius 2, then $\lim_{n \to +\infty} A(n) = 2\pi$.
2. If the area under the curve $y = x^2$ over an interval is approximated by the total area of a collection of rectangles, the approximation will be too large.
3. If $A(x)$ is the area under the graph of a nonnegative continuous function f over an interval $[a, x]$, then $A'(x) = f(x)$.
4. If $A(x)$ is the area under the graph of a nonnegative continuous function f over an interval $[a, x]$, then $A(x)$ will be a continuous function.

4.2 The Indefinite Integral

Name Group No.

KEYWORD: antiderivative, antidifferentiation or integration, indefinite integral, integrand, integral sign, constant of integration, differential equation, and slope field or direction field

Integration formulas

DIFFERENTIATION FORMULA	INTEGRATION FORMULA	DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x]=1$	$\int dx = x + C$	8. $\frac{d}{dx}[-\csc x]=\csc x\cot x$	$\int \csc x\cot x\,dx = -\csc x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right]=x^r \quad (r\neq -1)$	$\int x^r\,dx = \frac{x^{r+1}}{r+1}+C \quad (r\neq -1)$	9. $\frac{d}{dx}[e^x]=e^x$	$\int e^x\,dx = e^x + C$
3. $\frac{d}{dx}[\sin x]=\cos x$	$\int \cos x\,dx = \sin x + C$	10. $\frac{d}{dx}\left[\frac{b^x}{\ln b}\right]=b^x \quad (0<b,\, b\neq 1)$	$\int b^x\,dx = \frac{b^x}{\ln b}+C \quad (0<b,\, b\neq 1)$
4. $\frac{d}{dx}[-\cos x]=\sin x$	$\int \sin x\,dx = -\cos x + C$	11. $\frac{d}{dx}[\ln\lvert x\rvert]=\frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
5. $\frac{d}{dx}[\tan x]=\sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$	12. $\frac{d}{dx}[\tan^{-1} x]=\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
6. $\frac{d}{dx}[-\cot x]=\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$	13. $\frac{d}{dx}[\sin^{-1} x]=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
7. $\frac{d}{dx}[\sec x]=\sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + C$	14. $\frac{d}{dx}[\sec^{-1}\lvert x\rvert]=\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}\lvert x\rvert + C$

Properties of indefinite integrals

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

Find the following integrals. Check your answer by differentiating.

1. $\int (6x^2-4x+3)\,dx$

2. $\int (4x^3-7x^2+3x-5)\,dx$

3. $\int (2x^4+3x^3-5x^2-2x+1)\,dx$

4. $\int (2x^4+6x^3+7x^2-4x+1)\,dx$

.

5. $\int (5x^2 - 7x + 6)\, dx$

6. $\int (6x^3 - 6x^2 + 8x - 2)\, dx$

7. $\int (3x^{-2} + 4x^{-3} + 6)\, dx$

8. $\int (5x^{3/2} + 2x^{1/2} - 7x + 3)\, dx$

9. $\int (7x^{-3} + 3x^{-1/2} - 5x^{-3/2} - 6)\, dx$

10. $\int (3x^{-1} + 4e^{2x} + 5x - 6)\, dx$

12. $\int (2e^{-3x} - 4x^{-1} + 5)\, dx$

13. $\int (2e^{-x} + 3e^{x} + x^2 - 6)\, dx$

14. $\int \left(4\sqrt{x} + 3\sqrt[3]{x^2} - \frac{4}{x^2} - \frac{3}{x} + 2\right)\, dx$

15. $\int \left(\frac{5}{x^3} - \frac{2}{x} + \frac{3}{\sqrt{x}} + x + 1\right)\, dx$

16. $\int \left(\frac{1}{e^x} + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{\sqrt{x}} + 3\right)\, dx$

17. $\int \frac{x^2 + 5x + 1}{x}\, dx$

PRACTICE PROBLEMS: EXERCISE SET 4.2

37–40 True–False Determine whether the statement is true or false. Explain your answer.

37. If $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(x)\,dx = F(x) + C$$

38. If C denotes a constant of integration, the two formulas

$$\int \cos x\,dx = \sin x + C$$

$$\int \cos x\,dx = (\sin x + \pi) + C$$

are both correct equations.

39. The function $f(x) = e^{-x} + 1$ is a solution to the initial-value problem

$$\frac{dy}{dx} = -\frac{1}{e^x}, \quad y(0) = 1$$

40. Every integral curve of the slope field

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

is the graph of an increasing function of x.

Solve the initial-value problems.

43. (a) $\dfrac{dy}{dx} = \sqrt[3]{x},\ y(1) = 2$

(b) $\dfrac{dy}{dt} = \sin t + 1,\ y\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$

(c) $\dfrac{dy}{dx} = \dfrac{x+1}{\sqrt{x}},\ y(1) = 0$

44. (a) $\dfrac{dy}{dx} = \dfrac{1}{(2x)^3},\ y(1) = 0$

(b) $\dfrac{dy}{dt} = \sec^2 t - \sin t,\ y\left(\dfrac{\pi}{4}\right) = 1$

(c) $\dfrac{dy}{dx} = x^2\sqrt{x^3},\ y(0) = 0$

47–50 A particle moves along an s-axis with position function $s = s(t)$ and velocity function $v(t) = s'(t)$. Use the given information to find $s(t)$.

47. $v(t) = 32t;\ \ s(0) = 20$ **48.** $v(t) = \cos t;\ \ s(0) = 2$

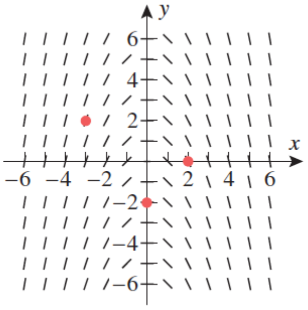
49. $v(t) = 3\sqrt{t};\ \ s(4) = 1$ **50.** $v(t) = 3e^t;\ \ s(1) = 0$

61–64 The given slope field figure corresponds to one of the differential equations below. Identify the differential equation that matches the figure, and sketch solution curves through the highlighted points.

(a) $\dfrac{dy}{dx} = 2$ (b) $\dfrac{dy}{dx} = -x$

(c) $\dfrac{dy}{dx} = x^2 - 4$ (d) $\dfrac{dy}{dx} = e^{x/3}$

61.



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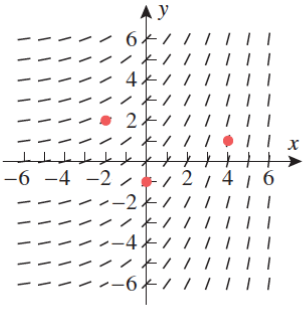

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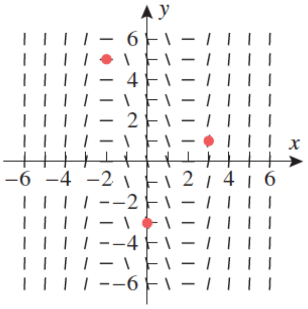

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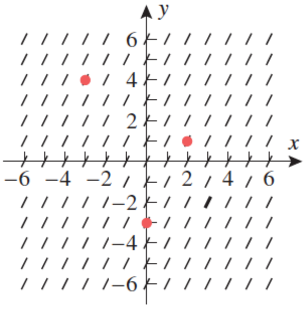

69–70 Use a trigonometric identity to evaluate the integral.

69. $\displaystyle\int \tan^2 x\, dx$ **70.** $\displaystyle\int \cot^2 x\, dx$

4.3 Integration by Substitution

Name	Group	No.

KEYWORD:

1. Evaluate the given integrals by making the given substitution.

   (a) $\int \cos 3x dx,\ u = 3x$

   (b) $\int x^2\sqrt{1+x^3}dx,\ u = 1 + x^3$

   (c) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx,\ u = \sqrt{x}$

   (d) $\int \cos t e^{\sin t} dt,\ u = \sin t$

   (e) $\int \frac{1}{\sqrt{5x+8}}dx,\ u = 5x + 8$

   (f) $\int \frac{1}{\sqrt{5x+8}}dx,\ u = \sqrt{5x+8}$

2. Evaluate $\int \sqrt{3+2x}dx$
3. Evaluate $\int \frac{1}{\sqrt{3x+5}}dx$
4. Evaluate $\int (3x+5)^{10}\, dx$
5. Evaluate $\int 2x\left(x^2+1\right)^4 dx$
6. Evaluate $\int \frac{(\ln x)^2}{x}dx$
7. Evaluate $\int \frac{dx}{3-5x}$
8. Evaluate $\int e^x\sqrt{1+e^x}dx$
9. Evaluate $\int \cos t \sin^4 t dt$
10. Evaluate $\int \frac{\sin 2x}{1+\cos^2 x}dx$
11. Evaluate $\int \frac{1+x}{1+x^2}dx$
12. Evaluate $\int \frac{dx}{x\ln x}$

The Definite Integral

(Section 4.4 -4.9)

Name Group No.

KEYWORD: Net Signed Area , Riemann sum or Riemann integral, The Definite Integral, dummy variable, and The Fundamental Theorem of Calculus

1. Evaluate

   a) $\int_{3}^{4} x+7x^2 dx$ b) $\int_{1}^{2} 6(x+\sqrt{x})dx$ c) $\int_{0}^{9} \frac{1}{2}\sqrt{x}dx$ d) $\int_{0}^{\pi/2} \sin x dx$

2. Evaluate $\int_{1}^{4} 3\sqrt{x}+\frac{1}{2\sqrt{x}}dx$

3. (evaluated by making the u-substitution)

   Find a) $\int_{-\pi/3}^{2\pi/3} \frac{\sin x}{\sqrt{2+\cos x}}dx$ b) $\int_{0}^{1} x^3\sqrt{x^2+3}dx$

4. **True–False**

Determine whether the statement is true or false. Explain your answer.

a) $\int_{0}^{6}(x^2+7)dx=\int_{0}^{3}(x^2+7)dx+\int_{3}^{6}(x^2+7)dx$

b) $\int_{2}^{2}(8+3x)dx=0$

c) $\int_{-1}^{1}\frac{1}{x}dx=0$

d) $\int_{1}^{10}5dx=50$

e) $\int_{-1}^{2}(9+7x-x^2)dx=\int_{-1}^{2}9dx+\int_{-1}^{2}7xdx-\int_{-1}^{2}x^2dx$

PRACTICE PROBLEMS: Textbook

29. A large juice glass containing 60 ml of orange juice is replenished by a server. The accompanying figure shows the rate at which orange juice is poured into the glass in milliliters per second (ml/s). Show that the average rate of change of the volume of juice in the glass during these 5 s is equal to the average value of the rate of flow of juice into the glass.

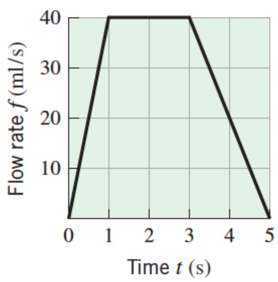

Figure Ex-29