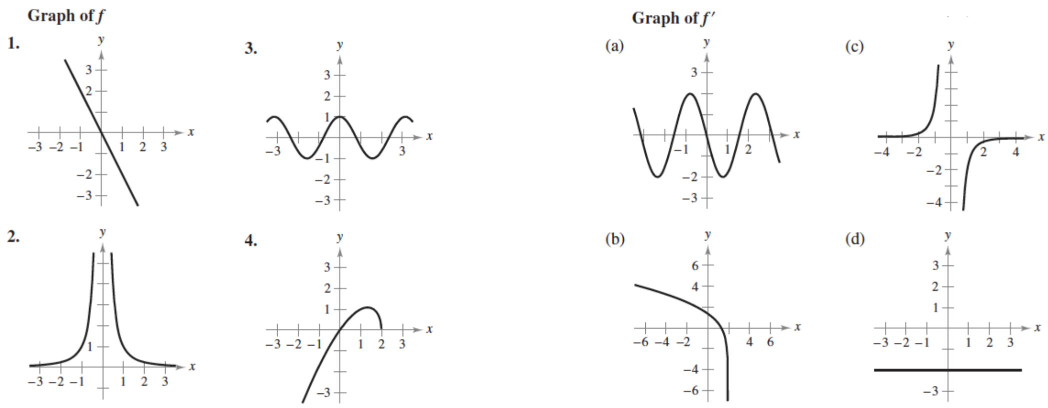
3.1 ANALYSIS OF FUNCTIONS I
INCREASE, DECREASE, AND CONCAVITY

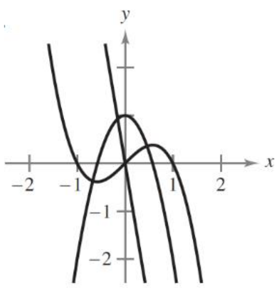
Name Group No.

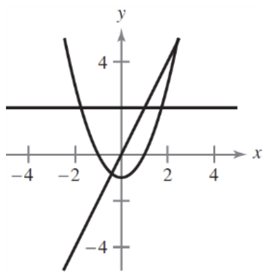
KEYWORD: increasing, decreasing, constant, concave up, concave down, and inflection point

Example 1. Match the graph of in the left column with that of its derivative in the right column.



Example 2. The graphs of f, f' and f'' are shown on the same set of coordinate axes. Which is which? Explain your reasoning.





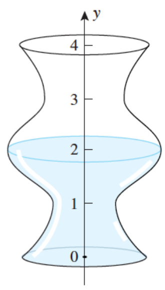
Example 3. Consider $f(x) = (\sqrt[3]{x^2} - 1)^2$

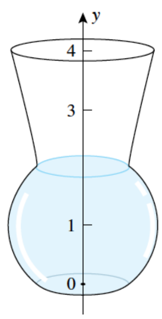
Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

Example 4. Consider $f(x) = \dfrac{x}{x^2 + 2}$

Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

Example 5. Suppose that water is flowing at a constant rate into the container shown. Make a rough sketch of the graph of the water level y versus the time t. Make sure that your sketch conveys where the graph is concave up and concave down, and label the y-coordinates of the inflection points.



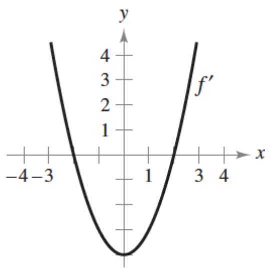


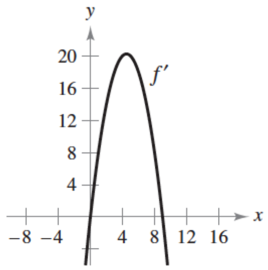
3.2 ANALYSIS OF FUNCTIONS II
RELATIVE EXTREMA; GRAPHING POLYNOMIALS

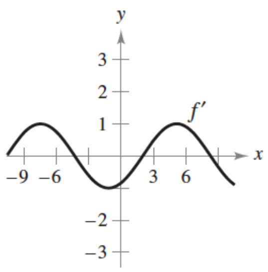
Name Group No.

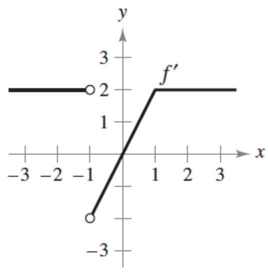
KEYWORD: critical point, stationary point, relative extremum (relative maximum and relative minimum), first derivative test, and second derivative test

Example 1. Use the graph of f' to sketch a graph of f and the graph of f''.

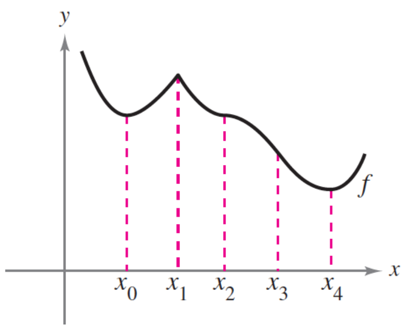








Example 2. Identify the real numbers x_1, x_2, x_3 and x_4 in the figure such that each of the following is true.



a. $f'(x) = 0$

b. $f''(x) = 0$

c. $f'(x)$ D.N.E.

d. f has a relative maximum.

e. f has a point of inflection.

Example 3. Sketch the graph of a function f with all of the following properties:

- domain of f is $\left[0,4\right]$
- $f(0)=1, f(1)=0, f(3)=2, f(4)=3$
- $f'(x)<0$ for $0<x<1$
- $f''(x)<0$ for $1<x<3$
- $f''(x)>0$ for $3<x<4$
- $\lim_{x\to 1^+} f'(x)=+\infty$ and $\lim_{x\to 1^-} f'(x)=-\infty$

Example 4. Locate the critical points and identify which critical points are stationary points.

1) $f(x)=4x^4-16x^2+17$

2) $f(x)=\dfrac{x^2}{x^3+8}$

3) $f(x)=\left|\sin(x)\right|$

3.3 ANALYSIS OF FUNCTIONS III

RATIONAL FUNCTIONS, CUSPS, AND VERTICAL TANGENTS

Name Group No.

KEYWORD: oblique or slant asymptote, curvilinear asymptote, and cusp

Example 1. Sketch the graph of the following and identify the locations of all relative extrema and inflection points.

1) $f(x) = \dfrac{x^2 + x}{1 - x^2}$

2) $f(x) = \dfrac{x}{x^2 - 4}$

3) $f(x) = \dfrac{x^2}{1 - x^3}$

4) $f(x) = \dfrac{(x-2)^3}{x^2}$

5) $f(x) = \dfrac{2x^3 + x^2 + 1}{x^2 + 1}$

6) $f(x) = \dfrac{x^3 + x^2 + 5}{x + 2}$

Example 2. Sketch the graph of the following.

1) $f(x) = x^{1/3}$

2) $f(x) = x^{1/4}$

3) $f(x) = x^{1/5}$

4) $f(x) = x^{2/5}$

5) $f(x) = x^{-1/3}$

6) $f(x) = \sqrt{x^2 - 4}$

7) $f(x) = \sqrt{|\, x \,|}$

8) $f(x) = x - \tan(x)$

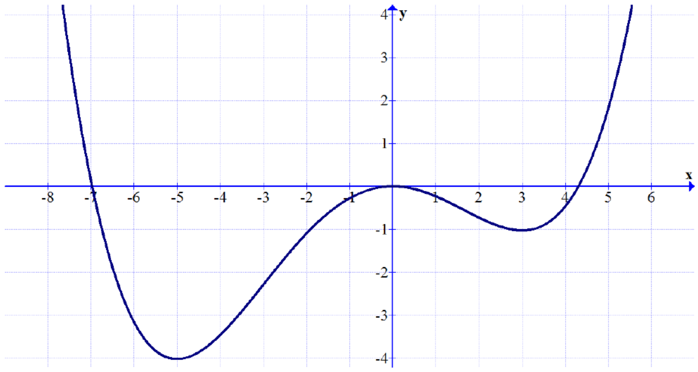
9) $f(x) = \sqrt{\tan(x)}$

3.4 ABSOLUTE MAXIMA AND MINIMA

Name Group No.

KEYWORD: absolute extremum (absolute maximum and absolute minimum), and Extreme-Value Theorem

Example 1. Consider the graph of $y = f(x)$, shown below. For each of the following, compute the absolute maximum and absolute minimum values of $f(x)$ on the given interval, if they exist. (Make reasonable assumptions about the behavior of the function outside of the shown interval.)



a. $(-\infty, \infty)$

b. $[-7, 5]$

c. $[-7, 0]$

d. $(-4, 1)$

Example 2. Sketch the graph of a continuous function, $y = f(x)$, which has all of the following properties:

- domain of f is $[1, 7]$.
- f has an absolute maximum of 6 when $x = 2$ and an absolute minumum of -1 when $x = 5$.
- $f''(x) > 0$ for all x in the domain of f, with the exception of $x = 2$ where $f''(x)$ DNE.

Example 3. For each of the following, find the absolute maximum and minimum values of $f(x)$ on the given interval.

$f(x)$	interval	absolute maximum value	absolute minimum value
a. $f(x) = x^3 + 3x - 4$	$[-3, 3]$		
b. $f(x) = (2x+1)^3$	$[-1, 4]$		
c. $f(x) = \dfrac{x-3}{(x-4)^2}$	$[-4, 1]$		
d. $f(x) = \cos(x) - \sin(x)$	$[-\pi, \pi]$		
e. $f(x) = \dfrac{x-2}{x+5}$	$(-\infty, \infty)$		
f. $f(x) = x^2 e^{-2x}$	$(-\infty, \infty)$		

3.5 APPLIED MAXIMUM AND MIMIMUM PROBLEMS

Name Group No.

KEYWORD: Use the techniques from Chapter 3.4 to solve optimization problems, i.e. given a system of related quantities, find values of the quantities that optimize one of them (e.g. minimize a cost, maximize a volume, etc)

PRACTICE PROBLEMS: EXERCISE SET 3.5

21. An open box is to be made from a 3 ft by 8 ft rectangular piece of sheet metal by cutting out squares of equal size from the four corners and bending up the sides. Find the maximum volume that the box can have.

23. A closed rectangular container with a square base is to have a volume of 2000 cm^3. It costs twice as much per square centimeter for the top and bottom as it does for the sides. Find the dimensions of the container of least cost.

31. A cylindrical can, open at the top, is to hold 500 cm^3 of liquid. Find the height and radius that minimize the amount of material needed to manufacture the can.

51. Find the coordinates of the point P on the curve

$$y = \frac{1}{x^2}, x > 0$$

where the segment of the tangent line at P that is cut off by the coordinate axes has its shortest length.