Solved Integration Problems

1 $\int \frac{dx}{\sqrt{x} - \sqrt[8]{x}}$

$$\int \frac{dx}{\sqrt{x} - \sqrt[8]{x}}$$

The computation of this integral combines the method of substitution and the method to integrate rational functions.

1st Step

Substitute $t = \sqrt[8]{x}, \; \sqrt{x} = t^4$

Compute
$$dt = \frac{dx}{8\left(\sqrt[8]{x}\right)^7} \Rightarrow dx = 8\left(\sqrt[8]{x}\right)^7 dt = 8t^7 dt$$

$$\int \frac{dx}{\sqrt{x} - \sqrt[8]{x}} = \int \frac{8t^7 dt}{t^4 - t} = \int \frac{8t^6 dt}{t^3 - 1}$$

Next we perform polynomial division

$$= \int \left(8t^3 + 8 + \frac{8}{t^3 - 1}\right) dt = 2t^4 + 8t + \int \frac{8dt}{t^3 - 1}$$

$$\int \frac{dx}{\sqrt{x} - \sqrt[8]{x}}$$

2nd Step **Partial Fraction Decomposition**

To compute $\int \frac{8dt}{t^3 - 1}$ apply partial fraction decomposition to the integrand $\frac{8}{t^3 - 1} = \frac{8}{(t-1)(t^2+t+1)}$

Compute

$$\frac{8}{(t-1)(t^2+t+1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+1}$$

$$\Leftrightarrow \frac{8}{(t-1)(t^2+t+1)} = \frac{A(t^2+t+1)}{(t-1)(t^2+t+1)} + \frac{(t-1)(Bt+C)}{(t-1)(t^2+t+1)}$$

$$\Leftrightarrow 8 = (A+B)t^2 + (A-B+C)t + A - C$$

$$\Leftrightarrow \begin{cases} A+B=0 \\ A-B+C=0 \\ A-C=8 \end{cases} \Leftrightarrow A = \frac{8}{3}, B = -\frac{8}{3}, C = -\frac{16}{3}$$

Next we have to integrate the partial fraction decomposition thus obtained.

$$\int \frac{dx}{\sqrt{x} - \sqrt[8]{x}}$$

3rd Step | **Integration of the Partial Fraction Decomposition**

Compute

$$\int \frac{8dt}{t^3 - 1} = \int \left(\left(\frac{8}{3(t-1)} \right) - \left(\frac{8t+16}{3(t^2+t+1)} \right) \right) dt$$

$$= \frac{8}{3} \ln|t-1| - \frac{8}{3} \int \frac{t+2}{t^2+t+1} dt$$

Use a substitution to compute this integral

$$\int \frac{t+2}{t^2+t+1} dt = \frac{1}{2} \int \frac{2t+1}{t^2+t+1} dt + \frac{3}{2} \int \frac{dt}{t^2+t+1}$$

$$= \frac{1}{2} \ln|t^2+t+1| + \frac{3}{2} \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{1}{2} \ln|t^2+t+1| + 2 \int \frac{dt}{\left(\frac{2t+1}{\sqrt{3}} \right)^2 + 1}$$

$$= \frac{1}{2} \ln|t^2+t+1| + \sqrt{3} \arctan\left(\frac{2t+1}{\sqrt{3}} \right) + C$$

Next: Combine the results

$$\int \frac{dx}{\sqrt{x}-\sqrt[8]{x}}$$

4th Step | Combine the Results

We have now obtained:

1 $$\int \frac{dx}{\sqrt{x}-\sqrt[8]{x}} = 2t^4 + 8t + \int \frac{8dt}{t^3-1}$$

2 $$\int \frac{8dt}{t^3-1} = \frac{8}{3}\ln|t-1| - \frac{8}{3}\int \frac{t+2}{t^2+t+1}dt$$

3 $$\int \frac{t+2}{t^2+t+1}dt = \frac{1}{2}\ln|t^2+t+1| + \sqrt{3}\arctan\left(\frac{2t+1}{\sqrt{3}}\right)+C$$

These formulae yield:

$$\int \frac{dx}{\sqrt{x}-\sqrt[8]{x}} = 2t^4 + 8t + \frac{8}{3}\ln|t-1| - \frac{4}{3}\ln|t^2+t+1| - \frac{8\sqrt{3}}{3}\arctan\left(\frac{2t+1}{\sqrt{3}}\right)+C$$

$$= 2\sqrt{x} + 8\sqrt[8]{x} + \frac{8}{3}\ln|\sqrt[8]{x}-1| - \frac{4}{3}\ln|\sqrt[4]{x}+\sqrt[8]{x}+1| - \frac{8\sqrt{3}}{3}\arctan\left(\frac{2\sqrt[8]{x}+1}{\sqrt{3}}\right)+C$$

$$\int \frac{dx}{\sqrt{x} - \sqrt[8]{x}}$$

Comment Manual computations have given the answer

$$\int \frac{dx}{\sqrt{x} - \sqrt[8]{x}} =$$

$$2\sqrt{x} + 8\sqrt[8]{x} + \frac{8}{3}\ln\left|\sqrt[8]{x} - 1\right| - \frac{4}{3}\ln\left|\sqrt[4]{x} + \sqrt[8]{x} + 1\right| - \frac{8\sqrt{3}}{3}\arctan\left(\frac{2\sqrt[8]{x} + 1}{\sqrt{3}}\right) + C$$

The same integral can be computed using Maple. The answer that Maple gives is, however, much more complicated. Next slide shows the result of the integration as computed by Maple 8.

$$\int \frac{dx}{\sqrt{x} - \sqrt[8]{x}}$$

Comment

Maple's answer is much more complicated:

$$\int \frac{1}{\sqrt{x} - x^{\left(\frac{1}{8}\right)}}\,dx = -\frac{1}{3}\sqrt{3}\arctan\left(\frac{(2\sqrt{x}+1)\sqrt{3}}{3}\right) + 2\sqrt{x} + 8x^{\left(\frac{1}{8}\right)} + \frac{4}{3}\ln\left(x^{\left(\frac{1}{8}\right)} - 1\right) - \frac{2}{3}\ln\left(x^{\left(\frac{1}{4}\right)} + x^{\left(\frac{1}{8}\right)} + 1\right)$$

$$-\frac{4}{3}\ln\left(x^{\left(\frac{1}{8}\right)} + 1\right) - \frac{4}{3}\left(\sum_{\_R = \mathrm{RootOf}(65536\,\_Z^8 - 256\,\_Z^4 + 1)} \_R\ln\left(x^{\left(\frac{1}{8}\right)} - 4\,\_R\right)\right) + \frac{1}{3}\ln(\sqrt{x} - 1)$$

$$+\frac{1}{6}\ln(x - \sqrt{x} + 1) - \frac{1}{3}\ln(\sqrt{x} + 1) - \frac{1}{6}\ln(x + \sqrt{x} + 1) + \frac{2}{3}\ln\left(x^{\left(\frac{1}{4}\right)} - x^{\left(\frac{1}{8}\right)} + 1\right) + \frac{1}{3}\ln(x - 1) - \frac{1}{6}\ln(x^2 + x + 1)$$

$$+\frac{1}{3}\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right) - \frac{1}{3}\sqrt{3}\arctan\left(\frac{(2\sqrt{x}-1)\sqrt{3}}{3}\right) + \frac{2}{3}\sqrt{3}\arctan\left(\frac{\left(2x^{\left(\frac{1}{4}\right)}+1\right)\sqrt{3}}{3}\right)$$

$$-\frac{4}{3}\sqrt{3}\arctan\left(\frac{\left(2x^{\left(\frac{1}{8}\right)}-1\right)\sqrt{3}}{3}\right) + \frac{2}{3}\sqrt{3}\arctan\left(\frac{\left(2x^{\left(\frac{1}{4}\right)}-1\right)\sqrt{3}}{3}\right) - \frac{1}{3}\ln\left(\sqrt{x} + x^{\left(\frac{1}{4}\right)} + 1\right) + \frac{2}{3}\ln\left(x^{\left(\frac{1}{4}\right)} - 1\right)$$

$$+\frac{1}{3}\ln\left(\sqrt{x} - x^{\left(\frac{1}{4}\right)} + 1\right) - \frac{2}{3}\ln\left(x^{\left(\frac{1}{4}\right)} + 1\right) - \frac{4}{3}\left(\sum_{\_R = \mathrm{RootOf}(65536\,\_Z^8 - 256\,\_Z^4 + 1)} \_R\ln\left(x^{\left(\frac{1}{8}\right)} + 4\,\_R\right)\right)$$

$$-\frac{4}{3}\sqrt{3}\arctan\left(\frac{\left(2x^{\left(\frac{1}{8}\right)}+1\right)\sqrt{3}}{3}\right)$$