Problems about Integration by Parts

1 $\int \arctan(7z)dz$

2 $\int x\arctan(x^2)dx$

3 $\int x^5\cos(x^3)dx$

4 $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx$

5 $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = 0$

6 $\int \sin(mx)\sin(nx)dx$

$\int \arctan(7z)dz$

Formula $$\int u dv = uv - \int v du$$

Computation Choose $u = \arctan(7z)$. Then

$$du = \frac{7}{1+(7z)^2}dz,\ dv = dz, \text{ and } v = z.$$

$$\int \arctan(7z)dz = z\arctan(7z) - \int \frac{7z}{1+(7z)^2}dz$$
$$= z\arctan(z) - \frac{1}{14}\ln\left|1+(7z)^2\right| + C.$$

The last integral has to be computed by the substitution $t = 1+(7z)^2$.

$$\int x \arctan(x^2) dx$$

Formula

$$\int u dv = uv - \int v du$$

Computation

Choose $u = \arctan(x^2)$. Then

$$du = \frac{2x dx}{1+x^4},\ dv = x dx,\ \text{and } v = \frac{x^2}{2}.$$

$$\int x \arctan(x^2) dx = \frac{1}{2} x^2 \arctan(x^2) - \int \frac{x^3}{1+x^4} dx$$

$$= \frac{1}{2} x^2 \arctan(x^2) - \frac{1}{4} \ln(1+x^4) + C$$

$\int x^5 \cos(x^3)dx$

Formula $\int u dv = uv - \int v du$

Computation Choose $u = x^3$. Then $du = 3x^2 dx$,

$dv = x^2 \cos\left(x^3\right)dx$, and $v = \dfrac{\sin\left(x^3\right)}{3}$.

To find v in the above we integrated $x^2\cos(x^3)$ by the substitution $t = x^3$.

$$\int x^5 \cos(x^3)dx = \frac{x^3}{3}\sin(x^3) - \int \frac{1}{3}\sin(x^3)3x^2 dx$$

$$= \frac{x^3}{3}\sin(x^3) - \int x^2 \sin(x^3)dx$$

Here we use the substitution $t = x^3$ again.

$$= \frac{x^3}{3}\sin(x^3) + \frac{1}{3}\cos(x^3) + C.$$

Show that $\int x^n e^x \, dx = x^n e^x - n\int x^{n-1} e^x \, dx$

Proof of the formula

Apply integration by parts setting $u = x^n$.

Then $du = nx^{n-1}dx$, and $dv = e^x \, dx$, $v = e^x$.

One gets

$$\int x^n e^x \, dx = x^n e^x - \int e^x \, nx^{n-1} dx$$

$$= x^n e^x - n\int x^{n-1} e^x \, dx.$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = 0$$

Problem

Show that $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = 0$

whenever $m^2 \neq n^2$, $m \neq 0$ and $n \neq 0$.

Justification

Two Integrations by Parts are needed.

Step 1

$$u = \sin(mx) \quad \Rightarrow dv = \sin(nx)dx,\ v = -\frac{1}{n}\cos(nx), du = m\cos(mx)dx.$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx \int \mathrm{dx} = -\frac{1}{n}\cos(nx)\sin(mx)\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(-\frac{1}{n}\cos(nx\right)m\cos(mx)dx$$

$$= \frac{m}{n}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = 0$$

Problem

Show that $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = 0$

whenever $m^2 \neq n^2$, $m \neq 0$ and $n \neq 0$.

Justification (cont'd)

Step 2

Next step is to compute $\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx$ by a second Integration by Parts.

Choose: $u = \cos(mx) \Rightarrow dv = \cos(nx)dx, v = \frac{1}{n}\sin(nx), du = m\sin(mx)dx.$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \frac{1}{n}\sin(nx)\cos(mx)\Big]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\left(\frac{1}{n}\sin(nx)\right)m\sin(mx)dx$$

$$= \frac{m}{n}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = 0$$

Problem

Show that $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = 0$

whenever $m^2 \neq n^2$, $m \neq 0$ and $n \neq 0$.

Justification (cont'd)

Step 3 Combining Step 1 and Step 2 we get

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \frac{m}{n}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \frac{m^2}{n^2}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx$$

Conclude

$$\left(1-\frac{m^2}{n^2}\right)\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0$$

By the assumptions, $1-\frac{m^2}{n^2} \neq 0$. Hence $\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0.$

$\int \sin(mx)\sin(nx)\,dx$

Problem Compute $\int \sin(mx)\sin(nx)\,dx$

Solution Use the formula

$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos((m-n)x) - \cos((m+n)x)]$$

To get

$$\int \sin(mx)\sin(nx)\mathrm{dx} = \frac{1}{2}\int[\cos((m-n)x) - \cos((m+n)x)]dx$$

$$= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\right] + C$$

By the FTC this result also justifies the formula of the previous example.

Assuming that $m \neq \pm n$.

If $m = n \neq 0$, then the above implies

$$\int \sin^2(mx)\mathrm{dx} = \frac{1}{2}\int[1-\cos(2mx)]dx = \frac{1}{2}\left[x - \frac{\sin(2mx)}{2m}\right] + C.$$