Solved Problems on Integration by Substitution

$$\int \sin(7z)dz$$

Substitute $t = 7z$

Compute $dt = 7dz \Rightarrow dz = \frac{dt}{7}$

$$\int \sin(7z)dz = \int \sin(t)\frac{dt}{7} = \frac{1}{7}\int \sin\left(t\right)dt$$

$$= -\frac{\cos\left(t\right)}{7} + C = -\frac{\cos\left(7z\right)}{7} + C$$

$$\int \sin(3\phi + 2)d\phi$$

Substitute $t = 3\phi + 2$

Compute $dt = 3d\phi \Rightarrow d\phi = \dfrac{dt}{3}$

$$\int \sin(3\phi + 2)d\phi = \int \sin(t)\frac{dt}{3} = \frac{1}{3}\int \sin(t)dt$$

$$= -\frac{\cos(t)}{3} + C = -\frac{\cos(3\phi + 2)}{3} + C$$

$$\int \frac{\tan^2(x)}{\cos^2(x)}dx$$

Recall

$$\frac{d\tan(x)}{dx} = \frac{1}{\cos^2(x)}$$

This differentiation formula suggests the correct substitution.

Substitute

$$t = \tan(x)$$

Compute

$$dt = \frac{dx}{\cos^2(x)}$$

$$\int \frac{\tan^2(x)}{\cos^2(x)}dx = \int t^2 dt$$

$$= \frac{t^3}{3} + C = \frac{\tan^3(x)}{3} + C$$

$$\int \frac{dx}{1+4x^2}$$

Rewrite

$$\int \frac{dx}{1+4x^2} = \int \frac{dx}{1+(2x)^2}$$

This rewriting helps us to find the correct substitution.

Substitute

$$t = 2x$$

Compute

$$dt = 2dx \Rightarrow dx = \frac{dt}{2}$$

$$\int \frac{1}{1+4x^2} dx = \int \frac{1}{1+t^2} \frac{dt}{2} = \frac{1}{2} \int \frac{dt}{1+t^2}$$

$$= \frac{1}{2} \arctan(t) + C = \frac{1}{2} \arctan(2x) + C$$

$$\int \frac{dx}{2+2x+x^2}$$

Rewrite

$$\int \frac{dx}{2+2x+x^2} = \int \frac{dx}{1+\left(1+2x+x^2\right)} = \int \frac{dx}{1+\left(x+1\right)^2}$$

Substitute

$$t = x+1$$

Compute

$$dt = dx$$

$$\int \frac{dx}{2+2x+x^2} = \int \frac{dt}{1+t^2}$$

$$= \arctan\left(t\right) + C = \arctan\left(x+1\right) + C$$

$$\int \sqrt{4x - x^2}\,dx$$

Rewrite $\int \sqrt{4x - x^2}\,dx = \int \sqrt{4 - \left(4 - 4x + x^2\right)}\,dx$

$$= \int \sqrt{4 - (x-2)^2}\,dx = \int 2\sqrt{1 - \left(\frac{x-2}{2}\right)^2}\,dx$$

Substitute $\frac{x-2}{2} = \sin(t)$

This rewriting helps to find the correct substitution.

Compute $dx = 2\cos(t)\,dt$

$$\int \sqrt{4x - x^2}\,dx = 4\int \sqrt{1 - \sin^2(t)}\cos(t)\,dt = 4\int \cos^2(t)\,dt$$

$$= 2\int \left(\cos(2t) + 1\right)dt = \sin(2t) + 2t + C = 2\sin(t)\cos(t) + 2t + C$$

$$= (x-2)\sqrt{1 - \left(\frac{x-2}{2}\right)^2} + 2\arcsin\left(\frac{x-2}{2}\right) + C$$

$$= \frac{(x-2)\sqrt{4x - x^2}}{2} + 2\arcsin\left(\frac{x-2}{2}\right) + C$$

$$\int \sqrt{4x^2+4x+10}\,dx$$

Rewrite

$$\int \sqrt{4x^2+4x+10}\,dx = \int \sqrt{9+\left((2x)^2+4x+1\right)}\,dx$$

$$= \int \sqrt{9+(2x+1)^2}\,dx = \int 3\sqrt{1+\left(\frac{2x+1}{3}\right)^2}\,dx$$

Substitute

$$\frac{2x+1}{3} = \sinh(t)$$

This rewriting helps to find the correct substitution.

Compute

$$dx = \frac{3}{2}\cosh(t)\,dt$$

$$\int \sqrt{4x^2+4x+10}\,dx = \frac{9}{2}\int \sqrt{1+\sinh^2(t)}\cosh(t)\,dt = \frac{9}{2}\int \cosh^2(t)\,dt$$

$$= \frac{9}{4}\int \left(\cosh(2t)+1\right)dt$$

Here we used formulae regarding hyperbolic functions

Substitute $s=2t$ to get

$$= \frac{9}{8}\sinh(2t)+\frac{9t}{4}+C = \frac{9}{4}\sinh(t)\cosh(t)+\frac{9t}{4}+C$$

$$\int \sqrt{4x^2+4x+10}\,dx$$

Substitute $\frac{2x+1}{3} = \sinh(t)$

Compute (cont'd)

Using the above substitution we concluded, on the previous slide that

$$\int \sqrt{4x^2+4x+10}\,dx = \frac{9}{4}\sinh(t)\cosh(t) + \frac{9t}{4} + C$$

Expressing $\cosh(t)$ in terms of $\sinh(t)$ we may return to the original variable:

$$\int \sqrt{4x^2+4x+10}\,dx = \frac{9}{4}\sinh(t)\sqrt{1+\sinh^2(t)} + \frac{9t}{4} + C$$

$$= \frac{6x+3}{4}\sqrt{1+\left(\frac{2x+1}{3}\right)^2} + \frac{9\sinh^{-1}\left(\frac{2x+1}{3}\right)}{4} + C$$

$$= \frac{2x+1}{4}\sqrt{4x^2+4x+10} + \frac{9\sinh^{-1}\left(\frac{2x+1}{3}\right)}{4} + C$$

This rewriting is essential in order to be able to cleanly express the result in the original variable.