Problems on Definite Integrals

Computing Riemann Sums
Using Riemann Sums
Estimating Integrals
Using the Fundamental Theorem of Calculus

Computing Riemann Sums

Problem 1

Use the definition of the integral to compute $\int_0^1 x^3 dx$.

Solution

$$\int_0^1 x^3 dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \left(\frac{1}{n}\right)$$

By the Definition

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3$$

Now use the formula for the sum of cubes

$$= \lim_{n\to\infty} \frac{1}{n^4} \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) = \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) = \frac{1}{4}.$$

Conclude

$$\int_0^1 x^3 dx = \frac{1}{4}.$$

Computing Riemann Sums

Problem 2

Compute left, right and middle point sums for the integral $\int_0^{\pi/2} \sin(x)\,dx$ with 5 subintervals.

Solution

The division points are $\{0, \pi/10, (2\pi)/10, (3\pi)/10, (4\pi)/10, \pi/2\}$.

$$\text{LEFT}(5) = \sin(0)\frac{\pi}{10} + \sin\left(\frac{\pi}{10}\right)\frac{\pi}{10} + \sin\left(\frac{2\pi}{10}\right)\frac{\pi}{10} + \sin\left(\frac{3\pi}{10}\right)\frac{\pi}{10} + \sin\left(\frac{4\pi}{10}\right)\frac{\pi}{10}$$

$$\approx 0.8347$$

$$\text{RIGHT}(5) = \sin\left(\frac{\pi}{10}\right)\frac{\pi}{10} + \sin\left(\frac{2\pi}{10}\right)\frac{\pi}{10} + \sin\left(\frac{3\pi}{10}\right)\frac{\pi}{10} + \sin\left(\frac{4\pi}{10}\right)\frac{\pi}{10} + \sin\left(\frac{\pi}{2}\right)\frac{\pi}{10}$$

$$\approx 1.1488$$

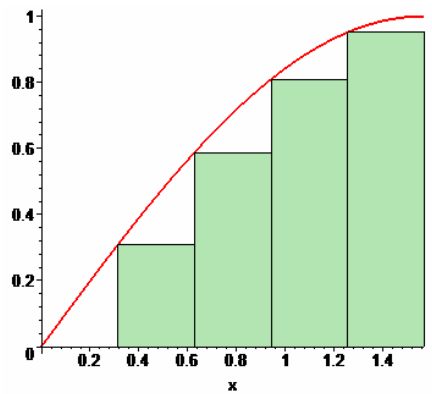
$$\text{MID}(5) = \sin\left(\frac{\pi}{20}\right)\frac{\pi}{10} + \sin\left(\frac{3\pi}{20}\right)\frac{\pi}{10} + \sin\left(\frac{5\pi}{20}\right)\frac{\pi}{10} + \sin\left(\frac{7\pi}{20}\right)\frac{\pi}{10} + \sin\left(\frac{9\pi}{20}\right)\frac{\pi}{10}$$

$$\approx 1.0041$$

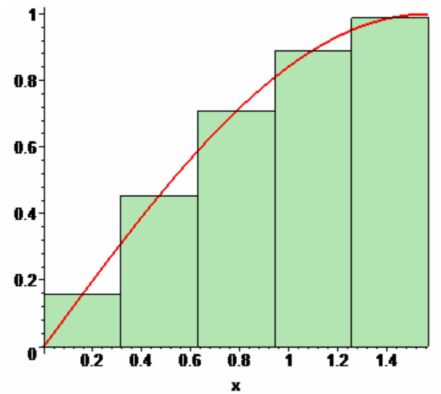
Computing Riemann Sums

Problem 2 Estimate the integral $\int_0^{\pi/2} \sin(x)\,dx.$

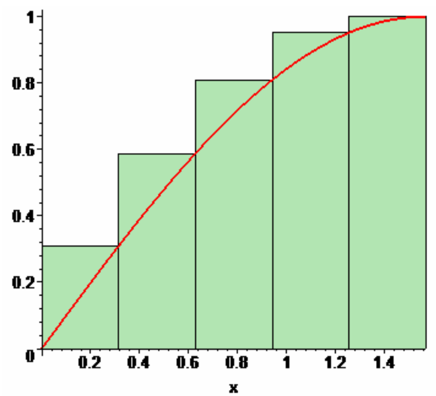
Comments on the Solution



LEFT(5)≈0.8347



MID(5)≈1.0041



RIGHT(5)≈1.1488

The function sin(x) is increasing on the interval of integration. Hence left sum gives a lower estimate for the integral and right sum an upper estimate. Middle sum estimate is the best, and not far from the actual value of the integral, which is 1 and can easily be computed by the Fundamental Theorem of Calculus.

Using Riemann Sums

Problem 3 Express the limit $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}$ as an integral.

Solution

The sum in question does not, at first sight, look like a Riemann sum for an integral because the summand is not of the form (function value)×(length of a subinterval). A rewriting is therefore necessary.

$$\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2} = \sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n}\left(\frac{1}{n}\right) = \sum_{k=1}^{n}\sqrt{\frac{n^2-k^2}{n^2}}\left(\frac{1}{n}\right)$$

$$= \sum_{k=1}^{n}\sqrt{1-\left(\frac{k}{n}\right)^2}\left(\frac{1}{n}\right) \xrightarrow[n\to\infty]{} \int_0^1 \sqrt{1-x^2}\,dx$$

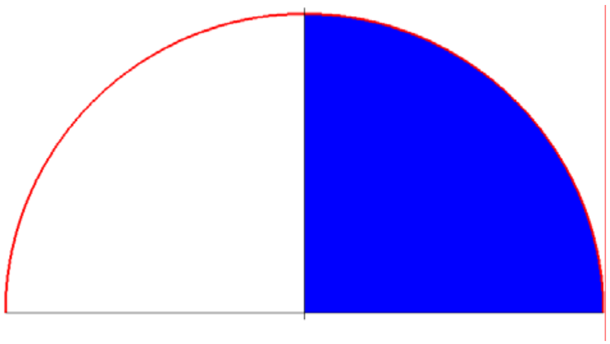
Now we have the length of a subinterval here.

Using Riemann Sums

Problem 3 Express the limit $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}$ as an integral.

Remarks The conclusion of the previous slide was that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}=\int_0^1\sqrt{1-x^2}\,dx.$$



To compute the value of this integral using the Fundamental Theorem of Calculus is a rather technical computation.

An easy way to compute the value of the integral is to observe that the graph of the function $\sqrt{1-x^2}$ is the upper half of a circle of radius 1.

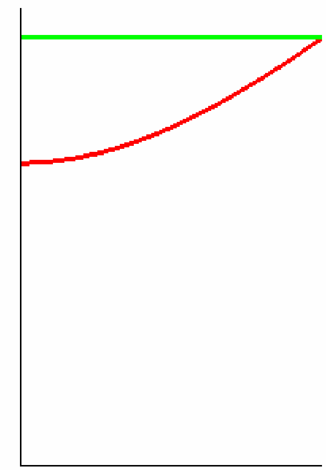
Hence the value of the integral is the area of the blue domain.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}=\int_0^1\sqrt{1-x^2}\,dx=\frac{\pi}{4}$$

Estimating Integrals

Problem 4 Show that $1 \le \int_0^1 \sqrt{1+x^2}\,dx \le \sqrt{2}.$

Solution The integral in question gives the area under the red graph and above the x-axis.



Observe that for $0 \le x \le 1, 1 \le \sqrt{1+x^2} \le \sqrt{2}.$

Hence the are of the domain under the blue line y=1 is a lower bound for the value of the integral.

This gives $1 \le \int_0^1 \sqrt{1+x^2}\,dx.$

The inequality $\int_0^1 \sqrt{1+x^2}\,dx \le \sqrt{2}$ follows from the fact that the domain under the red curve is contained in the rectangle under the green line $y = \sqrt{2}.$

Using the Fundamental Theorem

Problem 5 Let $F(x)=\int_{x}^{x^2}\frac{\sin(t)}{t}\,dt$. Compute $F'(x)$.

Solution The function F is defined in terms of an integral whose both lower and upper bound depends on x. This means that a rewriting is needed before we can apply the Fundamental Theorem of Calculus (FTC).

$$F(x)=\int_{0}^{x^2}\frac{\sin(t)}{t}\,dt-\int_{0}^{x}\frac{\sin(t)}{t}\,dt$$

The function $G(x)=\int_{0}^{x^2}\frac{\sin(t)}{t}\,dt$ is a composed function $U(V(x))$ where $U(v)=\int_{0}^{v}\frac{\sin(t)}{t}\,dt$ and $V(x)=x^2$.

By the Chain Rule.

By the FTC.

$$F'(x)=U'(V(x))V'(x)-\frac{\sin(x)}{x}=\frac{\sin(x^2)}{x^2}(2x)-\frac{\sin(x)}{x}=\frac{2\sin(x^2)-\sin(x)}{x}.$$

Using the Fundamental Theorem

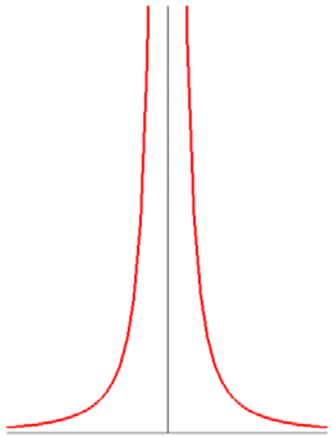
Problem 6

What is wrong in the following computation of $\int_{-1}^{1} \frac{1}{x^2}\,dx$?

Computation

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{1} x^{-2}\,dx = \left.\frac{x^{-2+1}}{-2+1}\right]_{-1}^{1}$$

$$= -\left.\frac{1}{x}\right]_{-1}^{1} = -\frac{1}{1} - \left(-\frac{1}{-1}\right) = -2.$$



This cannot be correct since the function to be integrated is positive. Hence the value of the integral should also be positive.

Answer

The graph of the function $1/x^2$ is shown in the figure above. The function is not continuous at $x=0$. Hence the integral cannot be computed by the FTC.

Using the Fundamental Theorem

Problem 7

The function f is continuous and

$$\int_0^{x^2} f(t)\,dt = e^{-x^2}.\quad \text{Determine } f(1).$$

Solution

We need to differentiate the given equation to get a formula for the function f.

Let $G(x) = \int_0^{x^2} f(t)\,dt$. The function $G(x)$ is a composed function

$U(V(x))$ where $U(v) = \int_0^{v} f(t)\,dt$ and $V(x) = x^2$.

So we get: $G'(x) = U'(V(x))V'(x) = f(x^2)2x$

Using FTC and the Chain Rule.

On the other hand: $G(x) = e^{-x^2}$. Hence $G'(x) = -2x\,e^{-x^2}$.

Conclude $2x\,f(x^2) = -2x\,e^{-x^2} \Rightarrow f(x^2) = -e^{-x^2}$ and $f(1) = -e^{-1}$.