Riemann Sums and Definite Integrals

Decompositions of Intervals

Riemann Sums

Definite Integrals

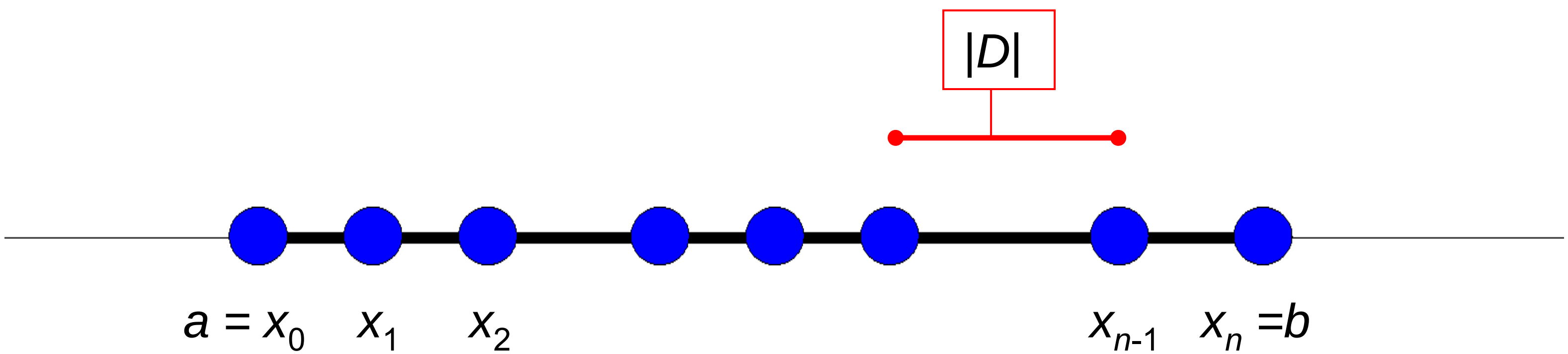
Numerical Integration

Divisions of Intervals to Subintervals

Definition

An ordered collection $D=(x_0,x_1,\ldots,x_n)$ of points of a closed interval $I = [a,b]$ satisfying $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ is a decomposition of the interval $[a,b]$ into subintervals $I_k=[x_{k-1},x_k]$.

For a decomposition $D=(x_0,x_1,\ldots,x_n)$, let $|D| = \max\{|x_k-x_{k-1}|, k=1,\ldots,n\}$. The quantity $|D|$ is the length of the longest subinterval I_k of the decomposition D.



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FAQ

Riemann Sums

Definition

Let f be defined on $[a,b]$, and let $D=(x_0,\ldots,x_n)$ be a decomposition of the interval $[a,b]$ into subintervals $I_k=[x_{k-1},x_k]$. Let $\zeta_k \in I_k \ \forall k=1,\ldots,n$. Let $\Delta x_k = x_k - x_{k\text{-}1}$. The sum

$$S_D=\sum_{k=1}^{n} f(\zeta_k)\Delta x_k$$

is a Riemann sum associated to the decomposition D.

Observe The Riemann sum S_D clearly depends on the choice of the points $\zeta_k \in I_k$.

Integrable Functions

Definition

A function f, defined on $[a,b]$, is integrable if the limit

$$\lim_{|D|\to 0} S_D = \lim_{|D|\to 0} \sum_{k=1}^{n} f(\zeta_k)\Delta x_k$$

is well defined, i.e., does not depend on the choice of the points $\zeta_k \in I_k$. The value of the above limit is the integral of f over the interval $[a,b]$.

Notation
$$\lim_{|D|\to 0} S_D = \lim_{|D|\to 0} \sum_{k=1}^{n} f(\zeta_k)\Delta x_k = \int_a^b f(x)dx$$

Theorem
All continous functions on a closed interval [a,b] are integrable.

Numerical Integration

The definition of the integral as a limit of Riemann sums allows us to approximate numerically definite integrals of integrable functions f. We have a freedom of choice in selecting the points ζ_k at which the function f needs to be evaluated. Depending on our selections we get different approximations.

Definition

We have the following approximations for the integral $\int_a^b \mathrm{f}(x)\,dx$:

1) the right approximation $\sum_{k=1}^{n} \mathrm{f}(x_k)\Delta x_k$

2) the left approximation $\sum_{k=1}^{n} \mathrm{f}(x_{k-1})\Delta x_k$

3) the mid-point approximation $\sum_{k=1}^{n} \mathrm{f}\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x_k$

These and refinements of these have been discussed in a Maple file.