Partial Fraction Decompositions

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Rational Functions

Definition A function of the type P/Q, where both P and Q are polynomials, is a rational function.

Example $\dfrac{x^3+1}{x^2+x+1}$ is a rational function.

The degree of the denominator of the above rational function is less than the degree of the numerator. Hence we may rewrite the above rational function in a simpler form by performing polynomial division.

Rewriting

$$\frac{x^3+1}{x^2+x+1} = x-1+\frac{2}{x^2+x+1}$$

For integration, it is always necessary to perform polynomial division first, if possible. To integrate the polynomial part is easy, and one can reduce the problem of integrating a general rational function to a problem of integrating a rational function whose denominator has degree greater than that of the numerator.

Example of Integrating Rational Functions

Integrating the functions $\dfrac{1}{1+x}$, $\dfrac{1}{1+x^2}$ and $\dfrac{2x}{1+x^2}$ is a simple task applying basic integration formulae and, in the last case, the substitution $u = 1 + x^2$.

One gets — here we omit the constants of integration —

$$\int \frac{1}{1+x}dx = \ln|1+x|,\ \int \frac{1}{1+x^2}dx = \arctan(x) \text{ and } \int \frac{2x}{1+x^2}dx = \ln(1+x^2)$$

Hence

$$\int \left(\frac{1}{1+x} + \frac{1}{1+x^2} + \frac{2x}{1+x^2} \right) dx = \ln|1+x| + \arctan(x) + \ln(1+x^2) + K.$$

I.e. $\displaystyle\int \frac{3x^2+3x+2}{x^3+x^2+x+1}dx = \ln|1+x| + \arctan(x) + \ln(1+x^2) + K.$

Here K is the constant of integration.

Partial Fraction Decomposition (1)

The integration

$$\int \frac{3x^2+3x+2}{x^3+x^2+x+1}dx = \ln|1+x| + \arctan(x) + \ln(1+x^2) + C$$

was based on the decomposition

$$\frac{3x^2+3x+2}{x^3+x^2+x+1} = \frac{1}{1+x} + \frac{1}{1+x^2} + \frac{2x}{1+x^2}$$

of the function to be integrated.

Definition The above decomposition of the rational function $(3x^2+3x+2)/(x^3+x^2+x+1)$ is a partial fraction decomposition.

Partial Fraction Decomposition is a rewriting of a rational function into a sum a rational functions with as simple denominators as possible.

General partial fraction decomposition is technically complicated and involves several cases. It all starts with a factorization of the denominator. The type of the partial factor decomposition depends on the type of the factors of the denominator. The different cases will be explained on the following slides.

Partial Fraction Decompositions (2)

The partial fraction decomposition of a rational function R=P/Q, with deg(P) < deg(Q), depends on the factors of the denominator Q. Since we are factoring over real numbers, the denominator Q may have following types of factors:

1. Simple, non-repeated linear factors $ax + b$.
2. Repeated linear factors of the form $(ax + b)^k$, $k > 1$.
3. Simple, non-repeated quadratic factors of the type $ax^2 + bx + c$. Since we assume that these factors cannot anymore be factorized, we have $b^2 - 4\,ac < 0$.
4. Repeated quadratic factors $(ax^2 + bx + c)^k$, $k>1$. Also in this case we have $b^2 - 4\,ac < 0$.

Finding the partial fraction decompositions is, in all of the above cases, same type of computation. Eventually one wants to integrate the resulting partial fraction decomposition. The integration can, in all cases, be based on formulae, but the computations will get technically complicated.

Simple Linear Factors (1)

Case I

Consider a rational function of the type

$$\frac{\mathrm{P}(x)}{\mathrm{Q}(x)} = \frac{\mathrm{P}(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)}$$

where $a_j \neq 0$ for all j, $\dfrac{b_i}{a_i} \neq \dfrac{b_j}{a_j}$ for $i \neq j$, and $\deg(\mathrm{P}) < n = \deg(\mathrm{Q})$.

Partial Fraction Decomposition: Case I

$$\frac{\mathrm{P}(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

for some uniquely defined numbers $A_k, k = 1,\ldots,n$.

Simple Linear Factors (2)

Example Consider the rational function $\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)}$.

By the result concerning Case I we can find numbers A and B such that

$$\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Compute these numbers in the following way

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \Leftrightarrow \frac{2}{x^2-1} = \frac{A(x+1)}{(x-1)(x+1)} + \frac{B(x-1)}{(x+1)(x-1)}$$

$$\Leftrightarrow \frac{0 \bullet x + 2}{x^2-1} = \frac{(A+B)x + (A-B)}{x^2-1} \Leftrightarrow \begin{cases} A+B=0 \\ A-B=2 \end{cases} \Leftrightarrow \begin{cases} A=1 \\ B=-1 \end{cases}.$$

To get the equations for A and B we use the fact that two polynomials are the same if and only if their coefficients are the same.

So the partial fraction decomposition is $\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$.

Simple Quadratic Factors (1)

Case II

Consider a rational function of the type $\frac{P(x)}{Q(x)}$, $\deg(P) < \deg(Q)$.

Assume that the denominator $Q(x)$ has a quadratic factor $ax^2 + bx + c$.

Partial Fraction Decomposition: Case II

The quadratic factor $ax^2 + bx + c$ of the denominator leads to a term of the type $\frac{Ax + B}{ax^2 + bx + c}$ in the partial fraction decomposition.

Simple Quadratic Factors (2)

Example

The rational function $\frac{3}{x^3-1}=\frac{3}{(x-1)(x^2+x+1)}$ has a term of the type $\frac{Ax+B}{x^2+x+1}$ in its partial fraction decomposition.

$$\frac{3}{x^3-1}=\frac{Ax+B}{x^2+x+1}+\frac{C}{x-1}\Leftrightarrow\frac{3}{x^3-1}=\frac{(Ax+B)(x-1)}{(x-1)(x^2+x+1)}+\frac{C(x^2+x+1)}{(x^2+x+1)(x-1)}$$

$$\frac{3}{x^3-1}=\frac{(A+C)x^2+(C+B-A)x+C-B}{x^3-1}\Leftrightarrow\begin{cases}A+C=0\\C+B-A=0\\C-B=3\end{cases}\Leftrightarrow\begin{cases}A=-1\\B=-2.\\C=1\end{cases}$$

To get these equations use the fact that the coefficients of the two numerators must be the same.

Hence

$$\frac{3}{x^3-1}=\frac{1}{x-1}-\frac{x+2}{x^2+x+1}$$

Repeated Linear Factors (1)

Case III

Consider a rational function of the type $\frac{P(x)}{Q(x)}$, $\deg(P) < \deg(Q)$.

Assume that the denominator $Q(x)$ has a repeated linear factor $(ax+b)^k, k > 1$.

Partial Fraction Decomposition: Case III

The repeated linear factor $(ax+b)^k$ of the denominator leads to terms of the type $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$ in the partial fraction decomposition.

Repeated Linear Factors (2)

Example The rational function $\frac{4x^2+4x-4}{x^3+x^2-x-1}=\frac{4x^2+4x-4}{(x-1)(x+1)^2}$ has

a partial fraction decomposition of the type $\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1}$.

$$\frac{4x^2+4x-4}{x^3+x^2-x-1}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1}\Leftrightarrow$$

$$\frac{A(x+1)(x-1)+B(x-1)+C(x+1)^2}{(x-1)(x+1)^2}=\frac{4x^2+4x-4}{x^3+x^2-x-1}$$

Equate the coefficients of the numerators.

$$\Leftrightarrow\frac{(A+C)x^2+(B+2C)x-A-B+C}{(x-1)(x+1)^2}=\frac{4x^2+4x-4}{x^3+x^2-x-1}\Leftrightarrow\begin{cases}A+C=4\\B+2C=4\\-A-B+C=-4\end{cases}$$

$$\Leftrightarrow\begin{cases}A=3\\B=2\\C=1\end{cases}$$

We get $\frac{4x^2+4x-4}{x^3+x^2-x-1}=\frac{3}{x+1}+\frac{2}{(x+1)^2}+\frac{1}{x-1}$.

Repeated Quadratic Factors (1)

Case IV

Consider a rational function of the type $\frac{P(x)}{Q(x)}$, $\deg(P) < \deg(Q)$.

Assume that the denominator $Q(x)$ has a repeated quadratic factor $(ax^2 + bx + c)^k, k > 1.$

Partial Fraction Decomposition: Case IV

The repeated quadratic factor $(ax^2 + bx + c)^k$ of the denominator leads to terms of the type $\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$ in the partial fraction decomposition.

Repeated Quadratic Factors (2)

Example $\dfrac{2x^4+3x^2-x}{x^5-x^4+2x^3-2x^2+x-1}=\dfrac{2x^4+3x^2-x}{(x-1)(x^2+1)^2}$ has a partial fraction decomposition of the type $\dfrac{A_1x+B_1}{x^2+1}+\dfrac{A_2x+B_2}{(x^2+1)^2}+\dfrac{C}{x-1}$.

$$\frac{A_1x+B_1}{x^2+1}+\frac{A_2x+B_2}{(x^2+1)^2}+\frac{C}{x-1}=$$

$$\frac{(A_1x+B_1)(x^2+1)(x-1)+(A_2x+B_2)(x-1)+C(x^2+1)^2}{(x-1)(x^2+1)^2}$$

This can also be computed by Maple command convert.

Computing in the same way as before one gets $A_1=B_1=A_2=C=1$, and $B_2=0$. Hence $\dfrac{2x^4+3x^2-x}{x^5-x^4+2x^3-2x^2+x-1}=\dfrac{x+1}{x^2+1}+\dfrac{x}{(x^2+1)^2}+\dfrac{1}{x-1}$.

Integrating Partial Fraction Decompositions (1)

Partial Fraction Decomposition is the main method to integrate rational functions. After a general partial fraction decomposition one has to deal with integrals of the following types. There are four cases. Two first cases are easy.

1. $$\int \frac{A}{ax+b}dx = \frac{A}{a}\ln|ax+b| + K$$

Here K is the constant of integration.

2. $$\int \frac{A}{(ax+b)^l}dx = \frac{A}{a}\frac{(ax+b)^{1-l}}{1-l} + K,\ l \neq 1.$$

Integrating Partial Fraction Decompositions (2)

In the remaining cases we have to compute integrals of the type:

3. $\displaystyle\int \frac{Ax+B}{ax^2+bx+c}dx$ and 4. $\displaystyle\int \frac{Ax+B}{\left(ax^2+bx+c\right)^l}dx,\ l>1$

In the case 3, observe that – since the denominator cannot be factored further – we have $4ac-b^2>0$. By suitable rewritings and substitutions we get:

$$\int \frac{Ax+B}{ax^2+bx+c}dx = \frac{A}{2a}\ln\left|ax^2+bc+c\right| + \frac{\left(2B-Ab/a\right)\arctan\left(\dfrac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}} + K$$

Here K denotes the constant of integration.

Integrating Partial Fraction Decompositions (3)

Integrals of the type $\int \frac{Ax+B}{\left(ax^2+bx+c\right)^l}dx,\ l>1,$ are the last case.

By repeated integrations by parts these integrals can eventually be computed by the formula for the integrals of type 3.

Theorem

All rational functions $f=\frac{P}{Q}$ can be integrated by Partial Fraction Decompositions provided that the polynomial Q can be factored.

Integration Algorithm

Integration of a rational function $f = P/Q$, where P and Q are polynomials can be performed as follows.

1. If $\deg(Q) \leq \deg(P)$, perform polynomial division and write $P/Q = S + R/Q$, where S and R are polynomials with $\deg(R) < \deg(Q)$. Integrate the polynomial S.
2. Factorize the polynomials Q and R. Cancel the common factors. Perform Partial Fraction Decomposition to the simplified form of R/Q.
3. Integrate the Partial Fraction Decomposition.

Examples (1)

Example 1 Compute $\int \frac{3}{x^3-1}dx$.

Observe $x^3-1=(x-1)(x^2+x+1)$. Hence

$$\frac{3}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$ for some numbers A,B and C.

To compute these numbers A,B and C we get

$$\frac{3}{x^3-1}=\frac{A(x^2+x+1)}{(x-1)(x^2+x+1)}+\frac{(Bx+C)(x-1)}{(x^2+x+1)(x-1)}$$

$$\Leftrightarrow \frac{3}{x^3-1}=\frac{(A+B)x^2+(A-B+C)x+A-C}{x^3-1} \Leftrightarrow \begin{cases} A+B=0 \\ A-B+C=0 \\ A-C=3 \end{cases} \Leftrightarrow \begin{cases} A=1 \\ B=-1. \\ C=-2 \end{cases}$$

Hence

$$\frac{3}{x^3-1}=\frac{1}{x-1}-\frac{x+2}{x^2+x+1}$$

Examples (2)

Example 1 (cont'd) Compute $\int \frac{3}{x^3-1}dx.$

By the previous computations we now have

$$\int \frac{3}{x^3-1}dx = \int \frac{1}{x-1}dx - \int \frac{x+2}{x^2+x+1}dx$$

$$= \ln|x-1| - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{3}{2}\int \frac{1}{x^2+x+1}dx$$

$$= \ln|x-1| - \frac{1}{2}\ln\left(x^2+x+1\right) - \frac{3}{2}\int \frac{1}{(x+1/2)^2+3/4}dx$$

$$= \ln|x-1| - \frac{1}{2}\ln\left(x^2+x+1\right) - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + K$$

Substitute $u=x^2+x+1$ in the first remaining integral and rewrite the last integral.

This expression is the required substitution to finish the computation.

Examples (3)

Example 2 Compute $\int \frac{x^3 - x + 2}{x^2 - 1} dx.$

We can simplify the function to be integrated by performing polynomial division first. This needs to be done whenever possible. We get:

$$\frac{x^3 - x + 2}{x^2 - 1} = x + \frac{2}{x^2 - 1}$$

Partial fraction decomposition for the remaining rational expression leads to

$$\frac{x^3 - x + 2}{x^2 - 1} = x + \frac{2}{x^2 - 1} = x + \frac{1}{x-1} - \frac{1}{x+1}$$

Now we can integrate

$$\int \frac{x^3 - 1 + 2}{x^2 - 1} dx = \int \left(x + \frac{1}{x-1} - \frac{1}{x+1} \right) dx$$

$$= \frac{x^2}{2} + \ln|x-1| - \ln|x+1| + K = \frac{x^2}{2} + \ln\left|\frac{x-1}{x+1}\right| + K.$$

Examples (4)

Example 3 Compute $\int \frac{3x^2+2}{x\left(x^2+2\right)}dx.$

Partial fraction decomposition in this case yields

$$\frac{3x^2+2}{x\left(x^2+2\right)} = \frac{1}{x} + \frac{2x}{x^2+2}$$

This is easy to integrate.

It is, however, easier to observe that the substitution

$$t = x\left(x^2+2\right), dt = \left(3x^3+2\right)dx$$

Leads to the result immediately

$$\int \frac{3x^2+2}{x\left(x^2+2\right)}dx = \int \frac{dt}{t} = \ln|t| + K = \ln\left|x\left(x^2+2\right)\right| + K.$$