Mean Value Theorem

Increasing Functions
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Mean Value Theorem

Increasing Functions (1)

Assume that the function f is everywhere increasing and differentiable.

Then $\forall h \neq 0 : \dfrac{f(x+h)-f(x)}{h} > 0$. It follows that $f'(x) = \lim_{h\to\infty} f(x) \geq 0.$

This follows from the fact that f is increasing.

Differentiability implies that the limit exists.

Theorem

Assume that the function f is differentiable at a point x_0

$f'(x_0) > 0$. Then $\exists \delta > 0$ such that

1) $f(x) > f(x_0)$ for $x_0 < x < x_0 + \delta$, and

2) $f(x) < f(x_0)$ for $x_0 - \delta < x < x_0$.

Increasing Functions (2)

Theorem Assume $f'(x_0) > 0$. $\exists \delta > 0$ such that

1) $f(x) > f(x_0)$ for $x_0 < x < x_0 + \delta$, and
2) $f(x) < f(x_0)$ for $x_0 - \delta < x < x_0$.

Proof By the definition of the derivative and using the fact that $f'(x_0) > 0$: $\exists \delta > 0$ such that

$$0 < |x - x_0| < \delta \Rightarrow \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < f'(x_0).$$

Hence $0 < |x - x_0| < \delta \Rightarrow \dfrac{f(x) - f(x_0)}{x - x_0} > 0.$

This implies that if $x - x_0$ is positive, then also $f(x) - f(x_0)$ is positive proving the first statement. The second statement follows similarly. ∎

Local Extreme Values

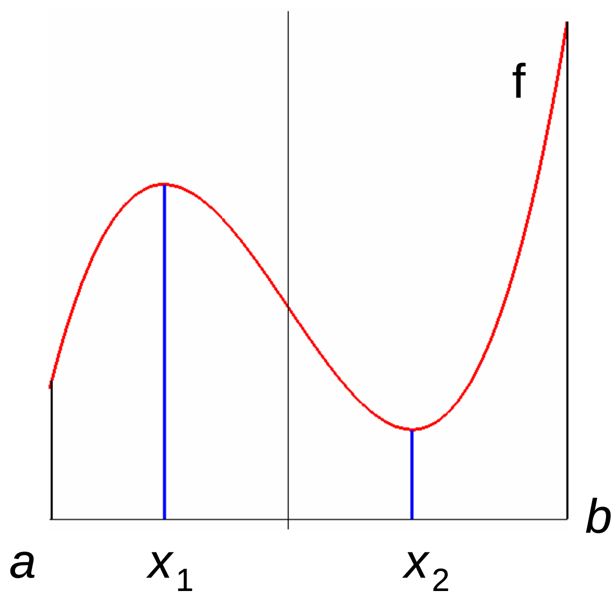
Definition

A point x_1 in the open interval (a,b) is a local maximum of the function f if, for values of x near the point p, $f(x) \leq f(x_1)$.

A point x_2 in the open interval (a,b) is a local minimum of the function f if, for values of x near the point p, $f(x) \geq f(x_2)$.

By a local extreme point of a function we mean either a local maximum or a local minimum. Values of a function at its extreme points are called extreme values.

In the figure on the right, the point x_1 is a local maximum and the point x_2 a local minimum of the function f.



Criteria for Local Extreme Values

Theorem Let f be a function which is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

If $c \in (a,b)$ is a local maximum or a local minimum of the function f, then the derivative of f vanishes at the point c, i.e. $f'(c)=0$.

Proof Assume that $c \in (a,b)$ is a local maximum or a local minimum of the function f.

If $f'(c) > 0$, then, by the previous Theorem, near the number c and on the right hand side of c, the function f takes values that are larger than $f(c)$. On the left hand side of the number c the function takes values which are smaller than $f(c)$. Hence c cannot be a local extreme value.

In the same way we see that $f'(c)$ cannot be negative. We conclude that $f'(c) = 0$. ∎

Extreme Values of Continuous Functions

Theorem

A function f which is continuous on a closed interval $[a,b]$ takes its maximum and minimum values on $[a,b]$.

We will not prove the result here. By geometric arguments the result appears plausible, and a rigorous proof uses arguments related to those used in the proof of the Intermediate Value Theorem for Continuous Functions.

Finding the Extreme Values

Let f be a function which is continuous on the closed interval [*a*,*b*] and differentiable on the open interval (*a*,*b*).

To find the extreme values of f on the interval [*a*,*b*], perform the following steps:

1. Compute the derivative of the function f.
2. Find the zeros of the derivative in the interval (*a*,*b*).
3. Compute the values of f at the zeros of the derivative and at the end-points *a* and *b*.
4. Among these computed values, choose the largest and the smallest. These are the extreme values of the function f.

Rolle's Theorem

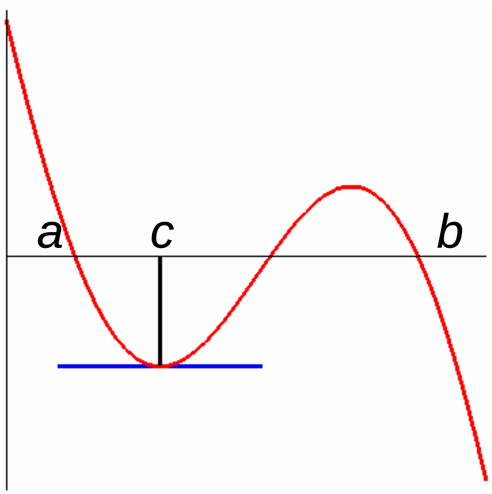
Theorem Let f be a function which satisfying:

1. f is continuous on the closed interval $[a,b]$,
2. f is differentiable on the open interval (a,b), and
3. $f(a)=f(b)$.

Then there is a point $c \in (a,b)$ such that the derivative of f vanishes at the point c, i.e., $f'(c) = 0$.

Rolle's Theorem Graphically

Rolle's Theorem states that, if $f(a) = f(b)$, then there is a point c between a and b such that the tangent of the graph of f at $(c,f(c))$ is horizontal.



Rolle's Theorem

Theorem

Let f be a function which satisfies the following conditions

1. f is continuous on the closed interval $[a,b]$,
2. f is differentiable on the open interval (a,b), and
3. $f(a)=f(b)$.

Then there is a point $c\in(a,b)$ such that the derivative of f vanishes at the point c, i.e., $f'(c) = 0$.

Proof

If $f(x)=f(a)=f(b)$ for all x between a and b, then f is a constant function, and the derivative of f vanishes for all x, and we can take c to be any point between a and b.

Rolle's Theorem

Theorem

Assume that f is continuous on [*a*,*b*] an differentiable on (*a*,*b*), and f(*a*)=f(*b*). Then there is a point $c \in (a,b)$ such that f'(*c*) = 0.

Proof (cont'd)

If f is not a constant function, then either its maximum on [*a*,*b*] is greater than f(*a*) or its minimum is smaller than f(*a*).

Assume that the maximum of f on [*a*,*b*] is greater than f(*a*). Then, since f(*b*) = f(*a*), f takes its maximum at a point $c \in (a,b)$.

By the Criteria for Local Extreme Values, we conclude that f'(*c*) = 0.

If the maximum of f on [*a*,*b*] is not creater than f(*a*), then the minium is smaller. And one can apply the Criteria for Local Extreme Values to the minimum of f to conclude the existence of *c*. ∎

The Mean Value Theorem

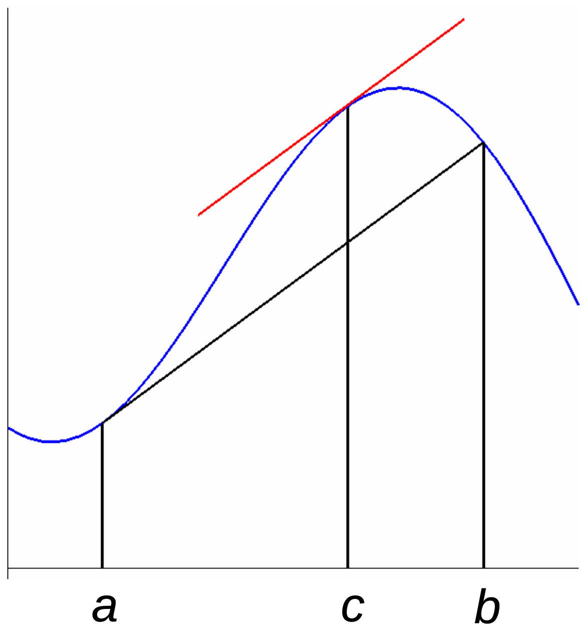
Theorem Let f be a function which satisfying:

1. f is continuous on the closed interval [*a*,*b*], and
2. f is differentiable on the open interval (*a*,*b*).

There is a point c, $a \le c \le b$, such that $f'(c) = (f(b) - f(a))/(b - a)$, i.e., $f(b) - f(a) = f'(c)(b - a)$.

Mean Value Theorem Graphically

The Mean Value Theorem states that between *a* and *b* there is a point *c* such that the tangent of the graph of f at (c, f(c)) is parallel to the line passing through (a,f(a)) and (b,f(b)).



The Mean Value Theorem

Theorem

Let f be a function which satisfies the following conditions

1. f is continuous on the closed interval $[a,b]$, and
2. f is differentiable on the open interval (a,b).

Then there is a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This condition can be expressed equivalently as

$$f(b) - f(a) = f'(c)(b - a).$$

Proof

Apply Rolle's Theorem to the function

$$g(x) = f(x) + (x - a)\frac{f(a) - f(b)}{b - a}$$

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FAQ

Corollary of the Mean Value Theorem

Theorem

Assume that the functions f and g are continous on $[a, b]$ and differentiable on (a, b). If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f - g$ is a constant function.

Proof Apply the Mean Value Theorem to the function h = f – g.

$\forall$ x_1 and x_2, $a \le x_1 < x_2 \le b, \exists\, c \in (x_1, x_2)$ such that

$$h(x_2) - h(x_1) = h'(c)(x_2 - x_1).$$

Since h'(c) = 0 by the assumptions and by the definition of the function h, $h(x_2) = h(x_1)$, i.e., h is a constant function. ■

Increasing Functions

Theorem

Assume that the function f is everywhere differentiable on an open interval (a,b), and that $\forall x \in (a,b): f'(x) > 0$. Then f is increasing on (a,b).

Proof

Let $x_1 < x_2$. We have to show that $f(x_1) < f(x_2)$.

By the Mean Value Theorem $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

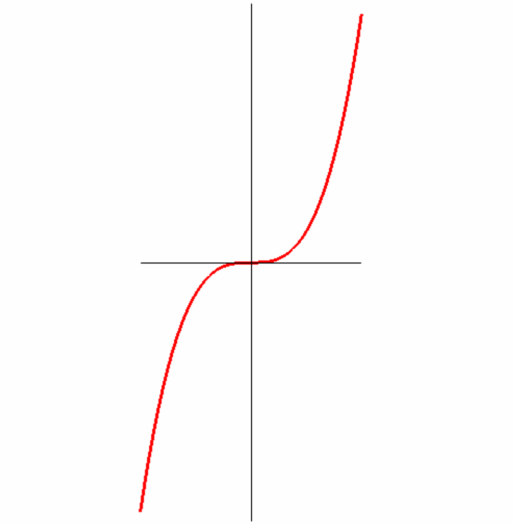
By the assumptions, $f'(c) > 0$. Hence also $f'(c)(x_2 - x_1) > 0$, i.e. $f(x_2) - f(x_1) > 0$. This means that $f(x_1) < f(x_2)$.

Increasing Functions (2)

Remark

The condition "$\forall x \in (a,b): f'(x) > 0$" of the previous theorem can be slightly relaxed: it is enough that $f'(x) > 0$ for all x except for a finite number of values of x.

$y = x^3$



Example

The function $f(x) = x^3$ is increasing even though $f'(x) = 3x^2 = 0$ for $x = 0$.

Decreasing Functions

Theorem

Assume that f is everywhere differentiable on (a, b) and that $f'(x) < 0$ for all x possibly excluding a finite number of values of x. Then the function f is decreasing on (a, b).

Proof

Let $g = -f$. Then the function f satisfies the conditions of the previous theorem, and g is increasing. Hence f is decreasing. ■