Integration by Parts

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Product Rule for Differentiation

Integration by Parts starts with the Product Rule for Differentiation:

$$\frac{d}{dx}(f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

This implies $f(x)g'(x) = \frac{d}{dx}(f(x)g(x)) - f'(x)g(x).$

Integrating both sides we get

$$\int f(x)g'(x)dx = \int \frac{d}{dx}\big(f(x)g(x)\big)dx - \int f'(x)g(x)dx$$

i.e.

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

Formula

$$\int udv = uv - \int vdu,$$

where $dv = g'(x)dx,\ u = f(x),\ du = f'(x)dx.$

Integration by Parts

Formula $$\int u\,dv = uv - \int v\,du$$

The idea is to use the above formula to simplify an integration task.

One wants to find a representation for the function to be integrated in the form *udv* so that the function *vdu* is easier to integrate than the original function.

Example $\int x\sin(x)dx.$ Choose $u = x.$ Then

$dv = \sin(x)dx,\ v = -\cos(x)$ and $du = dx.$

We get

$$\int x\sin(x)dx = -x\cos(x) - \int(-\cos(x))dx = -x\cos(x) + \int\cos(x)dx$$

$$= -x\cos(x) + \sin(x) + C.$$

In this example with the above choices, *vdu* = -cos(*x*)*dx,* which is easy to integrate. The choice *u*=sin(*x*) would have lead to a more complicated integral.

Examples (1)

Formula $$\int u\,dv = uv - \int v\,du$$

Example $$\int x^2 \cos(x)\,dx.$$

Choose $u = x^2$. Then $dv = \cos(x)dx,\ v = \sin(x),\ du = 2x\,dx$.

$$\int x^2 \cos(x)dx = x^2 \sin(x) - \int \sin(x)2x\,dx = x^2 \sin(x) - 2\int x \sin(x)dx$$

Repeat Integration by Parts for $\int x \sin(x)dx$ as on the previous slide.

One gets $\int x \sin(x)dx = -x\cos(x) + \sin(x) + C$. Hence

$$\int x^2 \cos(x)dx = x^2 \sin(x) - 2[-x\cos(x) + \sin(x)] + C \Rightarrow$$

$$= x^2 \sin(x) - 2x\cos(x) - 2\sin(x) + C.$$

Examples (2)

Formula $\int u\,dv = uv - \int v\,du$

Example $\int \sin^2 x\,dx.$

Choose $u = \sin(x),\ du = \cos(x)dx,\ dv = \sin(x)dx,\ v = -\cos(x).$

$$\int \sin^2(x)dx = -\cos(x)\sin(x) - \int \left(-\cos(x)\right)\cos(x)dx$$

$$= -\cos(x)\sin(x) + \int \cos^2(x)dx$$

$$\int \sin^2(x)dx = -\cos(x)\sin(x) + \int (1 - \sin^2(x))dx \Leftrightarrow$$

$$\int \sin^2(x)dx = -\cos(x)\sin(x) + \int dx - \int \sin^2(x)dx$$

$$2\int \sin^2(x)dx = -\cos(x)\sin(x) + x + C' \Rightarrow$$

$$\int \sin^2(x)dx = -\frac{1}{2}\cos(x)\sin(x) + \frac{x}{2} + C$$

In this case the integration problem was not simplified by Integration by Parts. We got, instead of a simplification, an equation from which we were able to solve the original integral.

Examples (3)

Formula $\int u\,dv = uv - \int v\,du$

Example $\int \arcsin x\,dx.$

Choose $u = \arcsin(x),\ du = \dfrac{dx}{\sqrt{1-x^2}},\ dv = dx,\ v = x.$

$$\int \arcsin(x)\,dx = x\arcsin(x) - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

To compute $\int \dfrac{x\,dx}{\sqrt{1-x^2}}$ use the substitution $t = 1-x^2, dt = -2x\,dx.$

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} + C = -\sqrt{1-x^2} + C.$$ This implies

$$\int \arcsin(x)\,dx = x\arcsin(x) - \left(-\sqrt{1-x^2}\right) + C$$
$$= x\arcsin(x) + \sqrt{1-x^2} + C$$

In this case the function to be integrated was not a product in any obvious way. This made it difficult to choose the term *dv*.

Examples (4)

Formula $\int u\,dv = uv - \int v\,du$

Example $\int \sin(x)e^x\,dx.$

Choose $u = \sin(x)$, $du = \cos(x)dx$, $dv = e^x\,dx$, $v = e^x$.

(1) $\int \sin(x)e^x\,dx = \sin(x)e^x - \int \cos(x)e^x\,dx$

Repeat Integration by Parts to compute $\int \cos(x)e^x\,dx$.

Choose $u = \cos(x), du = -\sin(x)dx, dv = e^x\,dx, v = e^x$.

(2) $\int \cos(x)e^x\,dx = \cos(x)e^x - \int(-\sin(x))e^x\,dx$

$(1) \wedge (2) \Rightarrow \int \sin(x)e^x\,dx = \sin(x)e^x - \left(\cos(x)e^x + \int \sin(x)e^x\,dx\right)$

$2\int \sin(x)e^x\,dx = \sin(x)e^x - \cos(x)e^x + C' \Rightarrow$

$\int \sin(x)e^x\,dx = \frac{1}{2}\sin(x)e^x - \frac{1}{2}\cos(x)e^x + C.$ Here $C = \frac{C'}{2}$.

In this case it is important to choose the term *du* in the second Integration by Parts correctly. The other obvious choice leads to the equation 0=0 which is not very useful.

Integration by Parts for Definite Integrals

Formula

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du = u(b)v(b) - u(a)v(a) - \int_a^b v\,du$$

Integration by Parts Formula and the Fundamental Theorem of Calculus imply the above Integration by Parts Formula for Definite Integrals. Here we must assume that the functions u and v and their derivatives are all continuous.

Example

$$\int_0^{1/2} \arcsin(x)\,dx$$

Choose $u = \arcsin(x)$. Then

$dv = dx$, $v = x$ and $du = \dfrac{dx}{\sqrt{1-x^2}}$.

$$\int_0^{1/2} \arcsin(x)\,dx = x\arcsin(x)\Big]_0^{1/2} - \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}}$$

To compute the last integral we still need to perform the substitution $t = x^2$.

Integration by Parts for Definite Integrals

Formula

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du = u(b)v(b) - u(a)v(a) - \int_a^b v\,du$$

Example (cont'd)

By the computations on the previous slide we now have

$$\int_0^{1/2} \arcsin(x)\,dx = x\arcsin(x)\Big]_0^{1/2} - \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} = \frac{\arcsin(1/2)}{2} - \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}}$$

Compute $\int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}}$ by the substitution $t = 1-x^2,\ dt = -2x\,dx.$

$x = 0 \Rightarrow t = 1$ and $x = \frac{1}{2} \Rightarrow t = \frac{3}{4}$

$$\int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} = \int_1^{3/4} \frac{-dt}{2\sqrt{t}} = -\sqrt{t}\Big]_1^{3/4} = -\frac{\sqrt{3}}{2} - (-1) = \frac{2-\sqrt{3}}{2}$$

Combining these results we get the answer

$$\int_0^{1/2} \arcsin(x)\,dx = \frac{\arcsin(1/2)}{2} - \frac{2-\sqrt{3}}{2} = \frac{\pi}{12} - \frac{2-\sqrt{3}}{2}$$